Optimization Theory and Methods

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- Multi-Stage Game With Perfect Information
- Two-Stage Quantity Competition
- Empty Threats
- Backward Induction
- Subgame Perfect PSNE
- Strategic Investment Example

Multi-Stage Game with Perfect Information

- A multi-stage game with perfect information is one where:
  1) All players know the actions chosen at all previous stages $0, 1, 2, \ldots, k-1$, when choosing their actions at stage k.
  2) In each stage k, some players take an action.
  3) All players who take an action in a particular stage take actions simultaneously in that stage.
- Thus, when a player takes an action in stage k, it is aware of all the actions taken by all players in the previous stages, but is not aware of the actions taken by other players in stage k.
- Important: A pure strategy is no longer the same as an action.
- A pure strategy is now a full contingency plan for all possible combinations of actions by all players at all previous stages.

- Consider a 2 player game. Similar to the Cournot case, the actions by the players are the choices of the output levels, q_1 for player 1 and q_2 for player 2.
  - We assume that at first player 1 chooses q_1, then player 2 chooses q_2.
  - Thus player 2 knows the value of q_1 before deciding q_2. So player 1's strategy is just a value of q_1, while player 2's strategy is a mapping giving <u>one value of q_2 for each value of q_1</u>.
  - Let $c = 0$ (for simplicity) and $p(q) = 12 - q$. So player i's payoff is $u_i(q_1, q_2) = [12 - (q_1 + q_2)] * q_i$.
  - So a PSNE of this game should include one number, q_1, and a mapping that gives **one value of $\boldsymbol{q_2}$ for each value of $\boldsymbol{q_1}$**.
  - How should we find this PSNE?

- The easy part is to compute the best q_2 value for each q_1 value.
- First order conditions give: $0 = \frac{\partial}{\partial q_2}[u_2(q_1, q_2)] = \frac{\partial}{\partial q_2}\big[[12 - (q_1 + q_2)]q_2\big] = 12 - q_1 - 2q_2$.
- So, $q_2 = 6 - \frac{q_1}{2}$. Thus, for any value of q_1 chosen by player 1, player 2 will observe that and accordingly pick a q_2 value equal to $6 - \frac{q_1}{2}$.
- So the equilibrium payoff of player 1 is given by: $u_1(q_1, q_2) = [12 - (q_1 + q_2)] * q_1 = \left[12 - \left(q_1 + 6 - \frac{q_1}{2}\right)\right] * q_1 = \left[6 - \frac{q_1}{2}\right] * q_1$.
- So the first order conditions give: $0 = \frac{\partial}{\partial q_1}\left[\left[6 - \frac{q_1}{2}\right] * q_1\right] = 6 - q_1$.
- So the PSNE is given by: $\boldsymbol{q_1{}^* = 6}$ **and** $\boldsymbol{q_2{}^*(q_1) = 6 - \frac{q_1}{2}}$.
- This PSNE is called a *Stackelberg Equilibrium*.

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- However, there are other PSNEs of this game!!
- For example, what if player 2 decides to have a strategy where it will pick a constant value $q_2 = {q_2}^C$ for any value q_1 picked by player 1. In other words, players 2 does not change its strategy in response to the observed value of q_1.
- In this case, $u_1(q_1, q_2) = [12 - (q_1 + q_2)] * q_1$. So 1st order conditions for player 1 give $0 = \frac{\partial}{\partial q_1}[u_1(q_1, q_2)] = \frac{\partial}{\partial q_1}\big[[12 - (q_1 + q_2)] * q_1\big] = 12 - 2q_1 - q_2$.
- Since q_2 does not depend on q_1, we get, $0 = \frac{\partial}{\partial q_2}[u_2(q_1, q_2)] = \frac{\partial}{\partial q_2}\big[[12 - (q_1 + q_2)] * q_2\big] = 12 - q_1 - 2q_2$.
- So, we get ${q_1}^* = 6 - \frac{{q_2}^C}{2}$ and ${q_2}^* = 6 - \frac{{q_1}^C}{2}$. Solving these simultaneously, we get ${q_1}^* = {q_2}^* = 4$.

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- In this new PSNE, player 1 assumes that player 2 will actually pick the exact same value of output q_2 even if player 1 picks a different value of q_1 (other than 4).
- Under this assumption, the best strategy for player 1 is to pick output $q_1 = 4$. But this is fully reliant on the assumption that player 1 actually believes that if it produces another level of output (e.g., $q_1 = 6$), player 2 will still play $q_2 = 4$ rather than playing $q_2 = 3$ which is the optimal response of player 2.
- In other words, player 2 threatens to player 1 that it will produce 4 units even though it is not optimal to player 2. This is what we call as empty threat. Note that $q_2 = 4$ is optimal for player 2 only when player 1 produces $q_1 = 4$. For all other levels of q_1, it is suboptimal.
- The PSNE on the previous slide assumes that player 1 will produce $q_1 = 4$ because it believes in player 2's threat.

- By assuming that player 1 will produce $q_1 = 4$, the response by player 2 to other levels of player 1 output does not matter.
- Yet, the very assumption that player 1 will produce $q_1 = 4$ is dependent on player 2 sticking to the strategy of always producing $q_2 = 4$ no matter what q_1 value is.
- As a result of the empty threat, neither player has a reason to deviate and hence this is a PSNE.
- However, this PSNE is highly suspect in terms of its portrayal of reality, because player 1 should be able to call player 2's "*bluff*" at some point.
- Hence, not all PSNEs are reasonable for a multi-stage game.
- We need a method to understand which equilibria are reasonable and which aren't.

- Consider how we calculated the Stackelberg equilibrium of the two-stage quantity competition game.
- Calculating the best strategy for player 2 in the second stage was relatively straightforward, because once q_1 was fixed, calculation of optimal ${q_2}^*$, as a function of q_1, was easy.
- Next we calculated the optimal q_1.
- So we started from the last stage of the game and worked backward to calculate the equilibrium of a multi-stage game.
- We will now extend the same idea to a general multi-stage game. This method of finding an equilibrium of a multi-stage game is called *backward induction*.
- For simplicity, we will only consider a situation where only one player takes an action at each stage of the game.

- Let h^k denote the set of actions actually taken by all players at all stages prior to stage k. Backward Induction algorithm works as follows:
  1) The algorithm begins by determining the optimal choices in the final stage K for each history h^K. This is determined by finding the action that maximizes the payoff of the player taking action at stage K contingent on a history h^K.
  2) Then it works back to stage $K-1$ and determines the optimal action for the player taking action in stage $K-1$, given that the player taking action in stage K with history h^K will take the action that we determined previously.
  3) We repeat this process one-by-one for stages $K-2$, $K-3$, ..., 2, 1.

- This process yields a PSNE where each player's actions are optimal for each possible action history (even those not taken).
- In other words, the problem of empty threats does not occur.
- This pure strategy Nash equilibrium is called a subgame perfect pure strategy Nash equilibrium.
- The idea of subgame perfection is not merely restricted to the multi-stage games with only one player taking action at each stage.
- We will now look at an example of subgame perfect PSNE for a more general multi-stage game where one or more players can take actions at each stage.

- Problem: Firm 1 and firm 2 both have a production cost of 2 per unit. Firm 1 can install a new technology which will make the production cost 0 per unit. But installing the technology will cost f. Firm 2 will observe whether or not firm 1 has installed the new technology. Then the two firms will simultaneously decide their outputs (q_1 and q_2). Assume that the price as a function of demand is $p(q_1, q_2) = 14 - (q_1 + q_2)$. Find the subgame perfect PSNE of this game.
- Solution: The payoff functions of the two players are:
  - $u_1(q_1, q_2) = \begin{cases} \big(12 - (q_1 + q_2)\big) * q_1 & \ldots \textit{if it doesn't invest} \\ \big(14 - (q_1 + q_2)\big) * q_1 - f & \ldots \textit{if invests in new tech} \end{cases}$
  - and $u_2(q_1, q_2) = \big(12 - (q_1 + q_2)\big) * q_2$.

- Now we will work backward to find a subgame perfect PSNE.
- If firm 1 does not invest in new technology, then we have the basic Cournot game that we saw earlier. The PSNE is obtained by solving the following two FOCs simultaneously: $q_1 = 6 - \frac{q_2}{2}$ and $q_2 = 6 - \frac{q_1}{2}$. As found earlier, the unique PSNE is at $(q_1, q_2) = (4{,}4)$. The payoff for each firm is 16 each.
- On the other hand, if firm 1 does invest in the new technology, then we will solve the following two FOCs simultaneously: $q_1 = 7 - \frac{q_2}{2}$ and $q_2 = 6 - \frac{q_1}{2}$. We get, $(q_1, q_2) = \left(\frac{16}{3}, \frac{10}{3}\right)$. Payoff of firm 1 is $\frac{256}{9} - f$. So firm 1 should make the investment if $\frac{256}{9} - f > 16$, i.e., if $f < \frac{112}{9}$.
- For example, if $f = 0$ then obviously firm 1 should invest in new technology, and if $f = 15$ then it shouldn't invest.

Objective :

Key Concepts :